B.Sc. Semester-VI Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 62111 **Course Code : SH/MTH/601/C-13**

Course Title : Metric Spaces and Complex Analysis

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

Answer **all** the questions.

1. Answer any **five** of the following questions: $2\times5=10$

a) Give an example of a metric space that is not separable, and explain why it fails to satisfy the separability property.

b) Let (X, d) be a complete metric space, and let A be a closed subset of X. Show that A is complete.

c) Let γ be the boundary of the unit disk centered at 0, traversed once in the counterclockwise direction. Compute the contour integral $\int_{\gamma} z^2 \sin\left(\frac{1}{z}\right) dz$.

[Turn Over]

d) Let $f(z)=\frac{1}{z^2+1}$. Find the Laurent series expansion of $f(z)$ about the point $z = i$, and use it to compute the residue of f(z) at $z = i$.

e) Let (X, d_n) be a usual metric space, where $X=\{x\in\mathbb{R}: x\geq 1\}$. Let a function $T: X\to X$ be such that $T(x)=\frac{x}{2}+\frac{1}{x}$, $x\in X$. Show that T is a contraction.

f) Show that $\left\{(x, y)\in\mathbb{R}^2 : x\neq 0,\ y=\sin\frac{1}{x}\right\}$ is a disconnected subset of the Euclidean space $\mathbb{R}^2$ with usual metric.

g) If $\lim_{z\to z_0} f(z)=\alpha\neq 0$, prove that there exists $\delta>0$ such that $|f(z)|>\frac{1}{2}|\alpha|$ for $0<|z-z_0|<\delta$.

h) Evaluate: $\oint_C \frac{\cos(e^z-1)}{z}dz$, where C represents the circle $|z|=2$ traversed once counter clockwise.

2. Answer any **four** of the following questions: $5\times4=20$

a) i) Prove that if (X, d) is a connected metric space, then any continuous function $f: X\to\{0, 1\}$ is constant.

ii) Show that if X is a compact metric space, then any sequence in X has a convergent subsequence. $2+3=5$

b) Let (X,d) be a compact metric space, and let $\{f_n\}$ be a sequence of continuous functions from X to R. If $\{f_n\}$ converges uniformly to a function f, show that f is also continuous. 5

c) i) Let A and B be two disjoint closed sets in a metric space (X, d), then show that there is a continuous function $f: X \to [0, 1]$ satisfying $f(a) = 0 \ \forall\, a \in A$ and $f(b) = 0 \ \forall\, b \in B$.

ii) Show that every countably compact metric space has Bolzano-Weierstrass property.

d) Let f(z) be an entire function such that $|f(z)| = M|z|^n$ for all z in C and some constants M and n. Show that f is a polynomial of degree at most n.

e) i) Let f be analytic in the domain $D = \{z \in \mathbb{C} : |z| < 2\}$. Prove that

$$2f(0) + f'(0) = \frac{2}{\pi}\int_0^{2\pi} f\left(e^{i\theta}\right)\cos^2\left(\frac{\theta}{2}\right)d\theta.$$

ii) Show that the function

$$f(z) = x^3 + 3ix^2y + axy^2 + by^3;$$

where a and b are complex constants, is analytic only if $a = -3$, $b = -i$.

f) Let C_R denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counter clockwise direction. Show that

$$\left|\int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}dz\right| \le \frac{\pi R\left(2R^2 + 1\right)}{\left(R^2 - 1\right)\left(R^2 - 4\right)}.$$

Hence show that the value of the integral tends to 0 as R tends to infinity.

3. Answer any **one** of the following questions: $10 \times 1 = 10$

a) i) Let (X, d) be a complete metric space and $f: X \to X$ be contraction map with Lipschitz constant K $(0 < K < 1)$. If $x_0 (\in X)$ is the unique fixed point of f, show that

$$d(x, x_0) \le \frac{1}{1-K} d(x, f(x)), \ \forall\, x \in X.$$

ii) A subset Γ of the real line $\mathbb{R}$, with at least two points is connected if Γ is an interval – prove it.

iii) Is the function $f(z) = xy + iy$ (x, y are reals) everywhere continuous? Is f(z) analytic? Justify. 4+3+3=10

b) i) Prove that a compact metric space is complete.

ii) Suppose that f(z) is entire and |f(z)| is bounded. Show that f(z) is constant.

iii) Use the Cauchy Integral Formula to compute the integral of $f(z) = \dfrac{2z+1}{z^2+z+1}$ around the positively oriented circle centered at the origin with radius 2. 4+3+3=10

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